Homework 5

Section 5.1

16. Prove that for every positive integer \( n \),

\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n + 1)(n + 2) = n(n + 1)(n + 2)(n + 3)/4.
\]

**Basis step:** Let \( n = 1 \). Then \( P(1) \)

\[
\sum_{i=1}^{1} i(i + 1)(i + 2) = 1 \cdot 2 \cdot 3 = 6
\]

and

\[
1(1 + 1)(1 + 2)(1 + 3)/4 = 1 \cdot 2 \cdot 3 \cdot 4/4 = 6
\]

**Inductive hypothesis:** Assume that for some positive integer \( k \) \( P(K) \) is

\[
\sum_{i=1}^{k} i(i + 1)(i + 2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) = k(k + 1)(k + 2)(k + 3)/4
\]

**Inductive step:** Prove \( P(K+1) \) is true based on \( P(k) \) is true. That is,

\[
\sum_{i=1}^{k+1} i(i + 1)(i + 2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)
\]

\[
= k(k + 1)(k + 2)(k + 3)/4 + (k + 1)(k + 2)(k + 3)
\]

\[
= (k + 1)(k + 2)(k + 3)(k + 4)/4
\]
20. Prove that $3^n < n!$ if $n$ is an integer greater than 6.

- **Basis Step:** Prove $P(7)$
  
  If $n = 7$, then $\text{LHS} = 3^7 = 2187$ and $\text{RHS} = 7! = 5040$.
  
  Hence, $\text{LHS} < \text{RHS}$.

- **Inductive Step**
  
  Assume that $3^n < n!$ for an arbitrary $n \geq 7$. – Induction Hypothesis.
  
  Prove this inequality hold for $n+1$ (ie. $3^{n+1} < (n+1)!$)
  
  $\text{LHS} = 3^{n+1} = 3^n.3$
  
  $\text{RHS} = (n+1)! = (n+1)n!$

  By using the induction hypothesis $3^n < n!$ and since $n$ is positive integer, we get $3^{n+1} < (n+1)n!$

  Since $(n+1)! = (n+1)n!$, we get $(n+1)3^n < (n+1)!$

  Since $n \geq 7$, $3 < (n+1)$.

  Therefore, $3^{n+1} < (n+1)3^n$

  Hence, $3^{n+1} < (n+1)!$
Bonus (5 points) Section 5.2

4. Let $P(n)$ be the statement that a postage of $n$ cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Solution

a) Show statement $P(18)$, $P(19)$, $P(20)$, $P(21)$ are true, completing the basis step of the proof.

$P(18)$: one 4-cent stamp and two 7-cent stamps.

$P(19)$: three 4-cent stamps and one 7-cent stamp.

$P(20)$: five 4-cent stamps.

$P(21)$: three 7-cent stamps.

b) What is the inductive hypothesis of the proof?

The inductive hypothesis is the statement that using just 4-cent and 8-cent stamps we can form $j$ cents postage for all $j$ with $18 \leq j \leq k$, where we assume that $k \geq 21$.

c) What do you need to prove in the inductive step?

We must show, assuming the inductive hypothesis, that we can form $k + 1$ cents postage using just 4-cent and 7-cent stamps.

d) Complete the inductive step for $k \geq 21$.

We want to form $k + 1$ cents of postage. Since $k \geq 21$, we know that $P(k - 3)$ is true, that is, that we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage, as desired.

e) We finished the proof of basis step and inductive step, based on the principles of strong induction, the statement $P(n)$ is true for all $n \geq 18$. 
Section 5.3

4. Find \( f(2), f(3), \) if \( f \) is defined recursively by \( f(0) = f(1) = 1 \) and for \( n = 1, 2, \ldots \)
   a) \( f(n + 1) = f(n) - f(n - 1). \)
   b) \( f(n + 1) = f(n) f(n - 1). \)

Solution:

(a) \( f(2) = f(1) - f(0) = 0 \)
    \( f(3) = f(2) - f(1) = -1 \)

(b) \( f(2) = f(1) * f(0) = 1 \)
    \( f(3) = f(2) f(1) = 1 \)

8. Give a recursive definition of the sequence \( \{a_n\}, n = 1, 2, 3, \ldots \) if
   a) \( a_n = 4n - 2. \)
   b) \( a_n = 1 + (-1)^n. \)

Solution:

(a) \( a_{n+1} = a_n + 4 \) or \( a_n = a_{n-1} + 4 \)

(b) \( a_{n+1} = a_n + (-2)(-1)^{n-1} \) or \( a_{n+1} = a_n + 2(-1)^n \)

24. Give a recursive definition of
   a) the set of odd positive integers.
   b) the set of positive integer powers of 3.

Solution:

(a) basis step and recursive step are as follows:
   
   \[
   1 \in S \\
   \text{If } x \in S, \text{ then } x + 2 \in S
   \]

(a) basis step and recursive step are as follows:
Section 5.4

8. Give a recursive algorithm for finding the sum of the first \( n \) positive integers.

Procedure Sum(n: first n positive integer)
if \( n=1 \) then Sum(n):=1
else Sum(n):= Sum(n-1) + n

44. Use a merge sort to sort 4, 3, 2, 5, 1, 8, 7, 6 into increasing order. Show all the steps used by the algorithm.