Homework 7

Section 7.1

22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

Solution: \( \frac{33}{100} \)

24. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 30.

Solution: \( \frac{1}{\binom{30}{6}} = \frac{1}{593775} \)

32. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

Solution: \( \frac{P(3,3)}{P(100,3)} = \frac{3 \times 2 \times 1}{100 \times 99 \times 98} \)
Section 7.2

6. What is the probability of these events when we randomly select a permutation of \{1, 2, 3\}? 
   a) 1 precedes 3.
   b) 3 precedes 1.
   c) 3 precedes 1 and 3 precedes 2.

Solution: (a)1/2  (b)1/2  (c)1/3

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Solution:
Sample space S: 2^5=32
E: exactly four heads appears: HHHHT, HHHTH, HHTHH, HTHHH, THHHH
F: first flip comes up tails: T*2^4=16
\(E \cap F = T H H H H\)
\(P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/32}{16/32} = 1/16\).

26. Let \(E\) be the event that a randomly generated bit string of length three contains an odd number of 1s, and let \(F\) be the event that the string starts with 1. Are \(E\) and \(F\) independent?

Solution:
Sample space S: 000, 001, 010, 011, 100, 101, 110, 111
E: 001, 010, 100, 111
F: 100, 101, 110, 111
\(E \cap F = \{100,111\}\)
\(P(E \cap F) = 2/8 = 1/4\)
\(P(E) = P(F) = 4/8 = 1/2\)
Since \(P(E) * P(F) = 1/2 * 1/2 = 1/4 = P(E \cap F)\)
Therefore, event E and F are independent.
Section 7.3

8. Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.

a) What is the probability that someone who tests positive has the genetic disease?

b) What is the probability that someone who tests negative does not have the disease?

Solution:

a) Let $E$ be the event that a person selected at random tests positive for the disease, and $F$ the event that a person selected at random has the disease.

$$p(F | E) = \frac{p(E | F) \cdot p(F)}{p(E | F) \cdot p(F) + p(E | \overline{F}) \cdot p(\overline{F})} = \frac{0.999 \cdot 0.0001}{0.999 \cdot 0.0001 + 0.0002 \cdot 0.9999} \approx 0.33,$$

so the probability that a person who tests positive has the disease is about 33%.

b) $E$ and $F$ as defined in a)

$$p(\overline{F} | \overline{E}) = \frac{p(\overline{E} | F) \cdot p(F)}{p(\overline{E} | F) \cdot p(F) + p(\overline{E} | F) \cdot p(\overline{F})} = \frac{0.9998 \cdot 0.9999}{0.9998 \cdot 0.9999 + 0.001 \cdot 0.001} \approx 0.9999999,$$

so the probability that a person who tests negative does not have the disease is about 99.99999%
Section 7.4

8. What is the expected sum of the numbers that appear when three fair dice are rolled?

Solution:

For a single die, the probabilities that the outcome is 1 or 2, or 3, ..., or 6 are the same, which is 1/6. So the expected value of the outcome is:

\[ E(n) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{1+2+3+4+5+6}{6} = 3.5 \]

For three independent fair dice the expected value of the sum of the outcomes is:

\[ E(n_1+n_2+n_3) = E(n_1) + E(n_2) + E(n_3) = 3.5 + 3.5 + 3.5 = 10.5 \]

10. Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

Solution:

There are 6 different outcomes of our experiment. Let the random variable \( X \) be the number of times we flip the coin. For \( i=1, 2, \ldots, 6 \), we need to compute the probability that \( X=i \). In order for this to happen when \( i<6 \), the first \( i-1 \) flips must contain exactly one tail, and there are \( i-1 \) ways this can happen. Therefore, \( p(X=i)=(i-1)/2^i \), since there are \( 2^i \) equally likely outcomes of \( i \) flips. So we have \( p(X=1)=0 \), \( p(X=2)=1/4 \), \( p(X=3)=2/8=1/4 \), \( p(X=4)=3/16 \), \( p(X=5)=4/32=1/8 \). To compute \( p(X=6) \), we note that this will happen when there is exactly one tail or no tails among the first five flips (probability \( 5/32+1/32=6/32=3/16 \)). As a check see that \( 0+1/4+1/4+3/16+1/8+3/16=1 \). We compute the expected number by summing \( i \) times \( p(X=i) \), so we get \( 1*0 + 2*1/4 + 3*1/4 + 4*3/16+5*1/8+6*3/16=3.75 \).

28. What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

Solution:
Let us call the outcome of a single roll a success if the number we get is 6, and call it a failure otherwise. We can see in this way that our experiment can be modeled as a sequence of 10 Bernoulli trials with probability of success equal to \( p = 1/6 \). The number of times a 6 appears is simply the number of successes in our outcome.

The variance can then be computed as:

\[
V \text{ (number of heads)} = 10 \cdot p \cdot (1 - p) = 10 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{18}.
\]