Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements
Example

We have the two premises:
- "if you have a current password, then you can log onto the network"
- "you have a current password"

And the conclusion:
- "you can log onto the network"

How do we get the conclusion from the premises?

The Argument

Argument: a sequence of statements that end with a conclusion

Former example:
- Premises: “if you have a current password, then you can log onto the network” and “you have a current password”
- Conclusion: “you can log onto the network”
  - p: “You have a current password”
  - q: “You can log onto the network.”

\[
\begin{array}{c}
p \rightarrow q \\
p \\
\hline
\therefore q
\end{array} \equiv (p \rightarrow q) \land p \rightarrow q
\]
Valid Arguments

- **Valid arguments:** the conclusion or final statement of the argument must follow from the truth of the preceding statements or premises of the argument.
  - the argument is true if and only if it is impossible for all the premises to be true and the conclusion to be false.

- The rules of inference are the essential building block in the construction of valid arguments.
  - Arguments in Propositional Logic
  - Arguments in Predicate Logic

Arguments in Propositional Logic

- **Definition 1:** An argument in propositional logic is a sequence of propositions. All but the final proposition are called premises. The last statement is the conclusion. The argument is valid if the premises imply the conclusion.

- An argument form in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter what propositions are substituted into its propositional variables in its premises, the conclusion is true if the premises are all true.
  - If the premises are \( p_1, p_2, ..., p_n \) and the conclusion is \( q \) then \( (p_1 \land p_2 \land ... \land p_n) \rightarrow q \) is a tautology.
**Rules of Inference for Propositional Logic**

**Modus ponens**

\[
\begin{align*}
  &p \\
  &p \rightarrow q \\
  &\therefore q
\end{align*}
\]

*Corresponding Tautology:*

\[
(p \land (p \rightarrow q)) \rightarrow q
\]

- **Example:**
  - \(p\): "It is snowing."
  - \(q\): "I will study discrete math."

- **Premises:**
  - "If it is snowing, then I will study discrete math."
  - "It is snowing."

- **Conclusion:**
  - "Therefore, I will study discrete math."

---

**Modus Tollens**

\[
\begin{align*}
  &\neg q \\
  &p \rightarrow q \\
  &\therefore \neg p
\end{align*}
\]

*Corresponding Tautology:*

\[
(\neg q \land (p \rightarrow q)) \rightarrow \neg p
\]

- **Example:**
  - \(p\) be "it is snowing."
  - \(q\) be "I will study discrete math."

- **Premises:**
  - "If it is snowing, then I will study discrete math."
  - "I will not study discrete math."

- **Conclusion:**
  - "Therefore, it is not snowing."
Hypothetical Syllogism

\[ p \rightarrow q \]
\[ q \rightarrow r \]
\[ \therefore p \rightarrow r \]

Corresponding Tautology:
\[(p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)\]

Example:
- Let \( p \) be “it snows.”
- Let \( q \) be “I will study discrete math.”
- Let \( r \) be “I will get an A.”

Premises:
- “If it snows, then I will study discrete math.”
- “If I study discrete math, I will get an A.”

Conclusion:
- “Therefore, if it snows, I will get an A.”

Resolution

\[ p \lor q \]
\[ \neg p \lor r \]
\[ \therefore q \lor r \]

Corresponding Tautology:
\[ ((\neg p \lor r) \land (p \lor q)) \rightarrow (q \lor r)\]

Example:
- Let \( p \) be “I will study discrete math.”
- Let \( r \) be “I will study English literature.”
- Let \( q \) be “I will study databases.”

Premises:
- “I will not study discrete math or I will study English literature.”
- “I will study discrete math or I will study databases.”

Conclusion:
- “Therefore, I will study databases or I will study English literature.”
**Rules of inference**

<table>
<thead>
<tr>
<th>Rule of Inference</th>
<th>Tautology</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( p )</td>
<td>Modus ponens</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>( p \rightarrow q )</td>
<td></td>
</tr>
<tr>
<td>( q )</td>
<td>( q )</td>
<td></td>
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</tbody>
</table>
| \( 
\begin{align*} \neg q \\ p \rightarrow q \end{align*} 
\) | \( (\neg q \rightarrow (p \rightarrow q)) \rightarrow \neg p \) | Modus tollens |
| \( p \rightarrow q \) | \( p \rightarrow q \) |              |
| \( q \rightarrow r \) | \( (p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \) | Hypothetical syllogism |
| \( \therefore p \rightarrow r \) | \( \therefore p \rightarrow r \) |              |
| \( p \lor q \) | \( p \lor q \) |              |
| \( \neg p \) | \( \neg p \) |              |
| \( \therefore q \lor r \) | \( (p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r) \) | Resolution |
| \( p \) | \( p \) |              |
| \( \therefore p \lor q \) | \( (p \lor q) \rightarrow p \) | Addition |
| \( p \land q \) | \( p \land q \) |              |
| \( \therefore p \lor q \) | \( (p \land q) \rightarrow (p \lor q) \) | Simplification |
| \( p \lor q \) | \( p \lor q \) |              |
| \( q \) | \( q \) |              |
| \( \therefore p \lor q \) | \( (p \lor q) \land q \rightarrow (p \lor q) \) | Conjunction |
| \( p \lor q \) | \( p \lor q \) |              |
| \( \neg q \lor r \) | \( (p \lor q) \land (\neg q \lor r) \rightarrow (q \lor r) \) | Resolution |

**Valid Arguments**

### Example 1: From the single proposition

\[ p \land (p \rightarrow q) \]

Show that \( q \) is a conclusion.

<table>
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<tr>
<th>Rule of Inference</th>
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<tr>
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\) | \( (\neg q \rightarrow (p \rightarrow q)) \rightarrow \neg p \) | Modus tollens |
| \( p \rightarrow q \) | \( p \rightarrow q \) |              |
| \( q \rightarrow r \) | \( (p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \) | Hypothetical syllogism |
| \( \therefore p \rightarrow r \) | \( \therefore p \rightarrow r \) |              |
| \( p \lor q \) | \( p \lor q \) |              |
| \( \neg p \) | \( \neg p \) |              |
| \( \therefore q \lor r \) | \( (p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r) \) | Resolution |
| \( p \land q \) | \( p \land q \) |              |
| \( \therefore p \lor q \) | \( (p \land q) \rightarrow (p \lor q) \) | Simplification |
| \( p \lor q \) | \( p \lor q \) |              |
| \( q \) | \( q \) |              |
| \( \therefore p \lor q \) | \( (p \lor q) \land (p \lor q) \rightarrow (p \lor q) \) | Conjunction |
| \( p \lor q \) | \( p \lor q \) |              |
| \( \neg p \lor r \) | \( (p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r) \) | Resolution |
Valid Arguments

Example 1: From the single proposition

\[ p \land (p \to q) \]

Show that \( q \) is a conclusion.

Solution:

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( p \land (p \to q) )</td>
<td>Premise</td>
</tr>
<tr>
<td>2. ( p )</td>
<td>Conjunction using (1)</td>
</tr>
<tr>
<td>3. ( p \to q )</td>
<td>Conjunction using (1)</td>
</tr>
<tr>
<td>4. ( q )</td>
<td>Modus Ponens using (2) and (3)</td>
</tr>
</tbody>
</table>

Valid Arguments

Example 2: With these hypotheses:

- “It is not sunny this afternoon and it is colder than yesterday.”
- “We will go swimming only if it is sunny.”
- “If we do not go swimming, then we will take a canoe trip.”
- “If we take a canoe trip, then we will be home by sunset.”

Using the inference rules, construct a valid argument for the conclusion:

“We will be home by sunset.”

Solution:

1. Choose propositional variables:
   - \( p \): “It is sunny this afternoon,” \( q \): “It is colder than yesterday,” \( r \): “We will go swimming,” \( s \): “We will take a canoe trip,” and \( t \): “We will be home by sunset.”

2. Translation into propositional logic:
   - Premises: \( \neg p \land q, r \to p, \neg r \to s, \text{ and } s \to t \)
   - Conclusion: \( t \)
Valid Arguments

- **Premises:** \( \neg p \land q, r \rightarrow p, \neg r \rightarrow s, \) and \( s \rightarrow t \)
- **Conclusion:** \( t \)

### Table for the Rules of Inference

<table>
<thead>
<tr>
<th>Rule of Inference</th>
<th>Tautology</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>((p \land (p \rightarrow q)) \rightarrow q)</td>
<td>Modus ponens</td>
</tr>
<tr>
<td>( \neg q \rightarrow \neg p )</td>
<td>((p \rightarrow q) \rightarrow \neg \neg q)</td>
<td>Modus tollens</td>
</tr>
<tr>
<td>( p \rightarrow q ) and ( q \rightarrow r )</td>
<td>((p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r))</td>
<td>Hypothetical syllogism</td>
</tr>
<tr>
<td>( p \lor q )</td>
<td>((p \lor q) \land \neg p \rightarrow q)</td>
<td>Disjunctive syllogism</td>
</tr>
<tr>
<td>( p \lor q \land \neg q \rightarrow p \lor q )</td>
<td>((p \lor q) \land \neg p \rightarrow q)</td>
<td>Adding</td>
</tr>
<tr>
<td>( p \land q \rightarrow p \lor q )</td>
<td>((p \land q) \rightarrow p)</td>
<td>Simplification</td>
</tr>
<tr>
<td>( p \land q \rightarrow (p \land q) \land q \rightarrow (p \land q) \land r \rightarrow (q \lor r) )</td>
<td>((p \land q) \land (q \lor r) \rightarrow (q \lor r) )</td>
<td>Resolution</td>
</tr>
</tbody>
</table>

Valid Arguments

- **p:** "It is sunny this afternoon,"  
  **q:** "It is colder than yesterday,"  
  **r:** "We will go swimming,"  
  **s:** "We will take a canoe trip,"  
  and **t:** "We will be home by sunset."
- **Premises:** \( \neg p \land q, r \rightarrow p, \neg r \rightarrow s, \) and \( s \rightarrow t \)
- **Conclusion:** \( t \)

3. **Construct the Valid Argument**

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \neg p \land q )</td>
<td>Premise</td>
</tr>
<tr>
<td>2. ( \neg p )</td>
<td>Simplification using (1)</td>
</tr>
<tr>
<td>3. ( r \rightarrow p )</td>
<td>Premise</td>
</tr>
<tr>
<td>4. ( \neg r )</td>
<td>Modus tollens using (2) and (3)</td>
</tr>
<tr>
<td>5. ( \neg r \rightarrow s )</td>
<td>Premise</td>
</tr>
<tr>
<td>6. ( s )</td>
<td>Modus ponens using (4) and (5)</td>
</tr>
<tr>
<td>7. ( s \rightarrow t )</td>
<td>Premise</td>
</tr>
<tr>
<td>8. ( t )</td>
<td>Modus ponens using (6) and (7)</td>
</tr>
</tbody>
</table>
Rules of Inference for predicate Logic

- **Universal Instantiation (UI)**
  \[
  \forall x P(x) \\
  \therefore P(c)
  \]

- **Example:**
  - Our domain consists of all dogs and Fido is a dog.
  **Premises:**
  - “All dogs are cuddly.”
  **Conclusion:**
  - “Therefore, Fido is cuddly.”

- **Universal Generalization (UG)**
  \[
  P(c) \text{ for an arbitrary } c \\
  \therefore \forall x P(x)
  \]

- **Existential Instantiation (EI)**
  \[
  \exists x P(x) \\
  \therefore P(c) \text{ for some element } c
  \]

- **Existential Generalization (EG)**
  \[
  P(c) \text{ for some element } c \\
  \therefore \exists x P(x)
  \]
Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that “John Smith has two legs” is a consequence of the premises: “Every man has two legs.” and “John Smith is a man.”

Solution:
- Let $M(x)$ denote “$x$ is a man” and $L(x)$ “$x$ has two legs” and let John Smith be a member of the domain.
- Valid Argument:

<table>
<thead>
<tr>
<th>TABLE 2</th>
<th>Rules of Inference for Quantified Statements.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule of Inference</td>
<td>Name</td>
</tr>
<tr>
<td>$\forall x \ P(x)$</td>
<td>$\therefore \ P(c)$</td>
</tr>
<tr>
<td>$P(c)$ for an arbitrary $c$</td>
<td>$\therefore \ \forall x \ P(x)$</td>
</tr>
<tr>
<td>$\exists x \ P(x)$</td>
<td>$\therefore \ P(c)$ for some element $c$</td>
</tr>
<tr>
<td>$P(c)$ for some element $c$</td>
<td>$\therefore \ \exists x \ P(x)$</td>
</tr>
</tbody>
</table>

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that “John Smith has two legs” is a consequence of the premises: “Every man has two legs.” and “John Smith is a man.”

Solution:
- Let $M(x)$ denote “$x$ is a man” and $L(x)$ “$x$ has two legs” and let John Smith be a member of the domain.
- Valid Argument:

Step | Reason
--- | ---
1. $\forall x (M(x) \rightarrow L(x))$ | Premise
2. $M(J) \rightarrow L(J)$ | UI from (1)
3. $M(J)$ | Premise
4. $L(J)$ | Modus Ponens using (2) and (3)
Example 2: Use the rules of inference to construct a valid argument showing that the conclusion
“Someone who passed the first exam has not read the book.”
follows from the premises
“A student in this class has not read the book.”
“Everyone in this class passed the first exam.”

Solution:
1. Let C(x) denote “x is in this class,” B(x) denote “x has read the book,” and P(x) denote “x passed the first exam.”
2. First we translate the premises and conclusion into symbolic form.
   \[
   \begin{align*}
   &\exists x (C(x) \land \neg B(x)) \\
   &\forall x (C(x) \to P(x)) \\
   \therefore &\exists x (P(x) \land \neg B(x))
   \end{align*}
   \]

Continued on next slide
Using Rules of Inference

Valid Argument:

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\exists x (C(x) \land \neg B(x))$</td>
<td>Premise</td>
</tr>
<tr>
<td>2. $C(a) \land \neg B(a)$</td>
<td>EI from (1)</td>
</tr>
<tr>
<td>3. $C(a)$</td>
<td>Simplification from (2)</td>
</tr>
<tr>
<td>4. $\forall x (C(x) \rightarrow P(x))$</td>
<td>Premise</td>
</tr>
<tr>
<td>5. $C(a) \rightarrow P(a)$</td>
<td>UI from (4)</td>
</tr>
<tr>
<td>6. $P(a)$</td>
<td>MP from (3) and (5)</td>
</tr>
<tr>
<td>7. $\neg B(a)$</td>
<td>Simplification from (2)</td>
</tr>
<tr>
<td>8. $P(a) \land \neg B(a)$</td>
<td>Conj from (6) and (7)</td>
</tr>
<tr>
<td>9. $\exists x (P(x) \land \neg B(x))$</td>
<td>EG from (8)</td>
</tr>
</tbody>
</table>

Combining rules of inference for propositions and quantified statements

- **Universal modus ponens (MP):** combines universal instantiation and modus ponens into one rule

$$\forall x (P(x) \rightarrow Q(x))$$

$P(a)$, where $a$ is a particular element in the domain

$\therefore Q(a)$
Returning to the Socrates Example

\[ \forall x (\text{Man}(x) \rightarrow \text{Mortal}(x)) \]
\[ \text{Man}(\text{Socrates}) \]
\[ \therefore \text{Mortal}(\text{Socrates}) \]

**Step**
1. \( \forall x (\text{Man}(x) \rightarrow \text{Mortal}(x)) \)
2. \( \text{Man}(\text{Socrates}) \rightarrow \text{Mortal}(\text{Socrates}) \)
3. \( \text{Man}(\text{Socrates}) \)
4. \( \text{Mortal}(\text{Socrates}) \)

**Reason**
Premise
UI from (4)
Premise
MP from (2) and (3)

Introduction to Proofs

Section 1.7
Methods of proving theorems

- Direct proofs
- Proof by contraposition
- Proofs by contradiction

Even and Odd Integers

- Definition: The integer \( n \) is even if there exists an integer \( k \) such that \( n = 2k \), and \( n \) is odd if there exists an integer \( k \), such that \( n = 2k + 1 \). Note that every integer is either even or odd and no integer is both even and odd.
Proving Conditional Statements: $p \rightarrow q$

- **Direct Proof**: Assume that $p$ is true. Use rules of inference, axioms and logical equivalences to show that $q$ must also be true.

- **Example**: Give a direct proof of the theorem “If $n$ is an odd integer, then $n^2$ is odd.”

- **Solution**:
  - Assume that $n$ is odd. Then $n = 2k + 1$ for an integer $k$. Squaring both sides of the equation, we get:
  - $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1$,
  - where $r = 2k^2 + 2k$, an integer.
  - We have proved that if $n$ is an odd integer, then $n^2$ is an odd integer.

(\n marks the end of the proof. Sometimes QED is used instead.)

---

Proving Conditional Statements: $p \rightarrow q$

- **Proof by Contraposition**: Assume $\neg q$ and show $\neg p$ is true also. This is sometimes called an indirect proof method. If we give a direct proof of $\neg q \rightarrow \neg p$ then we have a proof of $p \rightarrow q$.

- **Example**: Prove that if $n$ is an integer and $3n + 2$ is odd, then $n$ is odd.

- **Solution**:
  - **Contraposition**: if $n$ is even, then $3n+2$ is even
    - Assume $n$ is even. So, $n = 2k$ for some integer $k$. Thus $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2j$ for $j = 3k + 1$
    - Therefore $3n + 2$ is even. Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well. If $n$ is an integer and $3n + 2$ is odd (not even), then $n$ is odd (not even).
Proving Conditional Statements: \( p \rightarrow q \)

\* Proof by Contradiction: Suppose we want to prove that a statement \( p \) is true. Furthermore, suppose that we can find a contradiction \( q \) such that \( \neg p \rightarrow q \) is true. Because \( q \) is false, but \( \neg p \rightarrow q \) is true, we can conclude that \( \neg p \) is false, which means that \( p \) is true.

\* Example: Give a proof by contradiction of the theorem “If \( 3n + 2 \) is odd, then \( n \) is odd.”

\* Solution:
  
  - \( p: \text{"3n + 2 is odd"}, \quad q: \text{"n is odd."} \)
  
  - To construct a proof by contradiction, assume that both \( p \) and \( \neg q \) are true. That is, assume that \( 3n + 2 \) is odd and that \( n \) is not odd.
  
  - Because \( n \) is not odd, we know that it is even. Because \( n \) is even, there is an integer \( k \) such that \( n = 2k \). This implies that \( 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) \). Because \( 3n + 2 \) is \( 2t \), where \( t = 3k + 1 \), \( 3n + 2 \) is even. Note that the statement “\( 3n + 2 \) is even” is equivalent to the statement \( \neg p \), because an integer is even if and only if it is not odd. Because both \( p \) and \( \neg p \) are true, we have a contradiction. This completes the proof by contradiction, proving that if \( 3n + 2 \) is odd, then \( n \) is odd.
Reading assignment

- Section 1.8