Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
Sequences

Definition 1: A sequence is a function from a subset of the integers (usually either the set \{0, 1, 2, 3, 4, \ldots\} or \{1, 2, 3, 4, \ldots\}) to a set \(S\).

The notation \(a_n\) is used to denote the image of the integer \(n\). We can think of \(a_n\) as the equivalent of \(f(n)\) where \(f\) is a function from \{0,1,2,\ldots\} to \(S\). We call \(a_n\) a term of the sequence.

- A sequence is a discrete structure used to represent an ordered list.
- \(1,2,3,5,8\) is a finite sequence, \(1,3,9,27,\ldots,3^n,\ldots\) is an infinite sequence.

Example: consider the sequence \(\{a_n\}\), where \(a_n=1/n\), list the first four items of the sequence.

Solution:
- \(1,1/2,1/3,1/4\)

Geometric Progression

Definition 2: A geometric progression is a sequence of the form:
\[ a, ar, ar^2, \ldots, ar^n, \ldots \]
where the initial term \(a\) and the common ratio \(r\) are real numbers.

Examples:
1. Let \(a = 1\) and \(r = -1\). Then:
   \[\{b_n\} = \{b_0, b_1, b_2, b_3, \ldots\} = \{1, -1, 1, -1, \ldots\}\]
2. Let \(a = 2\) and \(r = 5\). Then:
   \[\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}\]
Arithmetic Progression

Definition 3: A arithmetic progression is a sequence of the form:

\[ a, a + d, a + 2d, \ldots, a + nd, \ldots \]

where the initial term \( a \) and the common difference \( d \) are real numbers.

Examples:
1. Let \( a = -1 \) and \( d = 4 \):
   \[ \{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\} \]
1. Let \( a = 1 \) and \( d = 2 \):
   \[ \{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\} \]

Recurrence Relations

Definition 4: A recurrence relation for the sequence \( \{a_n\} \) is a equation that expresses \( a_n \) in terms of one or more of the previous terms of the sequence, namely, \( a_0, a_1, \ldots, a_{n-1} \), for all integers \( n \) with \( n \geq n_0 \), where \( n_0 \) is a nonnegative integer.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

A recurrence relation is said to recursively define a sequence.

The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.
Questions about Recurrence Relations

Example 1: Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n = 1, 2, 3, 4, \ldots \) and suppose that \( a_0 = 2 \). What are \( a_1, a_2 \), and \( a_3 \)?

[Here \( a_0 = 2 \) is the initial condition.]

Solution:

We see from the recurrence relation that

\[
\begin{align*}
a_1 &= a_0 + 3 = 2 + 3 = 5 \\
a_2 &= 5 + 3 = 8 \\
a_3 &= 8 + 3 = 11
\end{align*}
\]

Questions about Recurrence Relations

Example 2: Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} - a_{n-2} \) for \( n = 2, 3, 4, \ldots \) and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2, a_3 \), and \( a_4 \)?

[Here the initial conditions are \( a_0 = 3 \) and \( a_1 = 5 \).]

Solution:

We see from the recurrence relation that

\[
\begin{align*}
a_2 &= a_1 - a_0 = 5 - 3 = 2 \\
a_3 &= a_2 - a_1 = 2 - 5 = -3 \\
a_4 &= a_3 - a_2 = -3 - 2 = -5
\end{align*}
\]
Fibonacci Sequence

- **Definition 5**: Define the Fibonacci sequence, \( f_0, f_1, f_2, \ldots \), by:
  - Initial Conditions: \( f_0 = 0, f_1 = 1 \)
  - Recurrence Relation: \( f_n = f_{n-1} + f_{n-2} \)
    for \( n = 2, 3, 4, \ldots \)

- **Example**: Find \( f_2, f_3, f_4, f_5 \) and \( f_6 \).
- **Answer**:
  - \( f_2 = f_1 + f_0 = 1 + 0 = 1 \),
  - \( f_3 = f_2 + f_1 = 1 + 1 = 2 \),
  - \( f_4 = f_3 + f_2 = 2 + 1 = 3 \),
  - \( f_5 = f_4 + f_3 = 3 + 2 = 5 \),
  - \( f_6 = f_5 + f_4 = 5 + 3 = 8 \).

Questions about Recurrence Relations

- **Example**: Determine whether the sequence \( \{a_n\} \), where \( a_n = 3n \) for every nonnegative integer \( n \), is a solution of the recurrence relation \( a_n = 2a_{n-1} - a_{n-2} \) for \( n = 2, 3, 4, \ldots \). Answer the same question where \( a_n = 2^n \).
- **Solution**:
  - Suppose that \( a_n = 3n \) for every nonnegative integer \( n \). Then, for \( n \geq 2 \), we see that \( 2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n \). Therefore, \( \{a_n\} \), where \( a_n = 3n \), is a solution of the recurrence relation.
  - Suppose that \( a_n = 2^n \) for every nonnegative integer \( n \). Note that \( a_0 = 1, a_1 = 2, \) and \( a_2 = 4 \). Because \( 2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2 \), we see that \( \{a_n\} \), where \( a_n = 2^n \), is not a solution of the recurrence relation.
Solving Recurrence Relations

- Finding a formula for the nth term of the sequence generated by a recurrence relation together with the initial conditions is called solving the recurrence relation.
- Such a formula is called a closed formula.

Iterative Solution Example

- Method 1: Working upward, forward substitution
- Starting with the initial condition, and working upward until we reach $a_n$ to deduce a closed formula for the sequence.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

\[ a_2 = 2 + 3 \]
\[ a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2 \]
\[ a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \]
\[ \vdots \]
\[ \vdots \]
\[ a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1) \]
Iterative Solution Example

- **Method 2:** Working downward, backward substitution
  - starting with the term $a_n$ and working downward until we reach the initial condition to deduce this same formula.

- Example: Let \{a_n\} be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

  \[
  a_n = a_{n-1} + 3 \\
  = (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
  = (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
  \vdots \\
  \vdots \\
  = a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1)
  \]

Special Integer Sequences

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.

- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?
Questions on Special Integer Sequences

Example 1: Find formulae for the sequences with the following first five terms: 1, 1/2, 1/4, 1/8, 1/16

Solution:
Note that the denominators are powers of 2. The sequence with \( a_n = \frac{1}{2^n} \) is a possible match. This is a geometric progression with \( a = 1 \) and \( r = \frac{1}{2} \).

Example 2: Consider 1, 3, 5, 7, 9

Solution:
Note that each term is obtained by adding 2 to the previous term. A possible formula is \( a_n = 2n + 1 \). This is an arithmetic progression with \( a = 1 \) and \( d = 2 \).

Useful Sequences

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>Some Useful Sequences.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^2 )</td>
<td>1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...</td>
</tr>
<tr>
<td>( n^3 )</td>
<td>1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...</td>
</tr>
<tr>
<td>( n^4 )</td>
<td>1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...</td>
</tr>
<tr>
<td>( 2^n )</td>
<td>2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...</td>
</tr>
<tr>
<td>( 3^n )</td>
<td>3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...</td>
</tr>
<tr>
<td>( n! )</td>
<td>1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...</td>
</tr>
<tr>
<td>( f_n )</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...</td>
</tr>
</tbody>
</table>
Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:
  $$\sum_{j=m}^{n} a_j, \quad \sum_{j=m}^{n} a_j, \quad \text{or} \quad \sum_{m \leq j \leq n} a_j$$
  (read as the sum from $j = m$ to $j = n$ of $a_j$)
  represents
  $$a_m + a_{m+1} + \cdots + a_n$$

- The variable $j$ is called the index of summation. It runs through all the integers starting with its lower limit $m$ and ending with its upper limit $n$.

Summations

- Example: What is the value of $\sum_{j=1}^{5} j^2$?
  - Solution:
    $$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
    $$= 1 + 4 + 9 + 16 + 25$$
    $$= 55.$$  

- Example: What is the value of $\sum_{k=4}^{8} (-1)^k$?
  - Solution:
    $$\sum_{k=4}^{8} (-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$
    $$= 1 + (-1) + 1 + (-1) + 1$$
    $$= 1.$$
Summations

- More generally for a set $S$:

$$
\sum_{j \in S} a_j
$$

- Example: What is the value of $\sum_{s \in \{0, 2, 4\}} s$?
  - Solution:
    - the sum of the values of $s$ for all the members of the set $\{0, 2, 4\}$,
    - \[
      \sum_{s \in \{0, 2, 4\}} s = 0 + 2 + 4 = 6.
    \]

Double summations

- Double summations arise in many contexts (as in the analysis of nested loops in computer programs)
  - first expand the inner summation and then continue by computing the outer summation

- Example:

$$
\sum_{i=1}^{4} \sum_{j=1}^{3} i j
$$

- Solution:

\[
\begin{align*}
\sum_{i=1}^{4} \sum_{j=1}^{3} ij &= \sum_{i=1}^{4} (i + 2i + 3i) \\
&= \sum_{i=1}^{4} 6i \\
&= 6 + 12 + 18 + 24 = 60.
\end{align*}
\]
<table>
<thead>
<tr>
<th>Sum</th>
<th>Closed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=0}^{n} ar^k, r \neq 0$</td>
<td>$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k$</td>
<td>$\frac{n(n + 1)}{2}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^2$</td>
<td>$\frac{n(n + 1)(2n + 1)}{6}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^3$</td>
<td>$\frac{n^2(n + 1)^2}{4}$</td>
</tr>
<tr>
<td>$\sum_{k=0}^{\infty} x^k,</td>
<td>x</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} kx^{k-1},</td>
<td>x</td>
</tr>
</tbody>
</table>