Chapter Summary

- 6.1 The Basics of Counting
- 6.2 The Pigeonhole Principle
- 6.3 Permutations and Combinations
- 6.5 Generalized Permutations and Combinations
The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- Examples, Examples, and Examples
The Product Rule

- **The Product Rule:** A procedure can be broken down into a sequence of two tasks. There are \( n_1 \) ways to do the first task and \( n_2 \) ways to do the second task. Then there are \( n_1 \cdot n_2 \) ways to do the procedure.

- **Example:** How many bit strings of length seven are there?
  - **Solution:** Since each of the seven bits is either a 0 or a 1, the answer is \( 2^7 = 128 \).

- **Example:** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?
  - **Solution:** By the product rule,
    - there are \( 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000 \) different possible license plates.

| 26 choices for each letter | 10 choices for each digit |
Telephone Numbering Plan

- Example: The North American numbering plan (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits:
  - Let X denote a digit from 0 through 9.
  - Let N denote a digit from 2 through 9.
  - Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was NYX-NNX-XXXX.
- In the new plan, the format is NXX-NXX-XXXX.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.
- There are 8 \cdot 2 \cdot 10 = 160 area codes with the format NYX.
- There are 8 \cdot 10 \cdot 10 = 800 area codes with the format NXX.
- There are 8 \cdot 8 \cdot 10 = 640 office codes with the format NNX.
- There are 10 \cdot 10 \cdot 10 \cdot 10 = 10,000 station codes with the format XXXX.

Number of old plan telephone numbers: 160 \cdot 640 \cdot 10,000 = 1,024,000,000.
Number of new plan telephone numbers: 800 \cdot 800 \cdot 10,000 = 6,400,000,000.

The Sum Rule

- The Sum Rule: If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways is the same as any of the \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution:
- By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.
Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

\[26 + 26 \cdot 10 = 286\]

Counting Passwords

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:

Let \( P \) be the total number of passwords, and let \( P_6, P_7, \) and \( P_8 \) be the passwords of length 6, 7, and 8.

By the sum rule \( P = P_6 + P_7 + P_8 \).

To find each of \( P_6, P_7, \) and \( P_8 \), we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

\[ P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560. \]
\[ P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920. \]
\[ P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880. \]

Consequently, \( P = P_6 + P_7 + P_8 = 2,684,483,063,360. \)
**Subtraction Rule**

- **Subtraction Rule:** If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, then the total number of ways to do the task is \( n_1 + n_2 \) minus the number of ways to do the task that are common to the two different ways.

- Also known as, the principle of **inclusion-exclusion:**

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

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**Counting Bit Strings**

- **Example:** How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

- **Solution:** Use the subtraction rule.
  - Number of bit strings of length eight that start with a 1 bit: \( 2^7 = 128 \)
  - Number of bit strings of length eight that start with bits 00: \( 2^6 = 64 \)
  - Number of bit strings of length eight that start with a 1 bit and end with bits 00: \( 2^5 = 32 \)

- Hence, the number is \( 128 + 64 - 32 = 160 \).
The Pigeonhole Principle
Section 6.2

Section Summary
- The Pigeonhole Principle
- The Generalized Pigeonhole Principle
The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

- Pigeonhole Principle: If \( k \) is a positive integer and \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.

- Proof by contradiction

Pigeonhole Principle

- Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
The Generalized Pigeonhole Principle

- **The Generalized Pigeonhole Principle:** If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

- **Example:** Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

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Example:

- a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- b) How many must be selected to guarantee that at least three hearts are selected?

Solution:

- a) Assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer $N$ such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.
- b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)
33. How many strings of eight English letters are there
   a) that contain no vowels, if letters can be repeated?
      \[ 21^8 \]
   b) that contain no vowels, if letters cannot be repeated?
      \[ 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \]
   c) that start with a vowel, if letters can be repeated?
      \[ 5 \times 21^8 \]
   d) that start with a vowel, if letters cannot be repeated?
      \[ 5 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \]
   e) that contain at least one vowel, if letters can be repeated?
      \[ 26^8 - 21^8 \]
   f) that contain exactly one vowel, if letters can be repeated?
      \[ 8 \times 5 \times 21^7 \]
   g) that start with X and contain at least one vowel, if letters can be repeated?
      \[ 1 \times 26^7 - 1 \times 21^7 \]
   h) that start and end with X and contain at least one vowel, if letters can be repeated?
      \[ 1 \times 26^6 \times 1 - 1 \times 21^6 \times 1 \]