Chapter Summary

- 7.1 Discrete Probability
- 7.2 Probability Theory
- 7.3 Bayes' Theorem
- 7.4 Expected value and Variance
Finite probability

- **Experiment**: is a procedure that yields one of a given set of possible outcomes.
- **Sample space of the experiment**: is the set of possible outcomes.
- **Event**: is a subset of the sample space.

If $S$ is a finite nonempty sample space of equally likely outcomes, and $E$ is an event, that is, a subset of $S$, then the **probability** of $E$ is $p(E) = |E| / |S|$. 
Finite probability

- What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

- Solution:
  - Sample space: all outcomes of the sum of the numbers on the two dice (1,1),(1,2),.....(6,5),(6,6)
  - Event: two dice with the sum of the numbers on the two dice is 7
    (1,6),(2,5),(3,4),(4,3),(5,2),(6,1)
  - Probability: \( p = \frac{6}{36} = \frac{1}{6} \)

Finite probability

- In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? what is the probability that a player wins the smaller prize?

- Solution1:
  - Only one way to choose all four digits correctly.
  - By the product rule, there are \( 10 \times 10 \times 10 \times 10 = 10,000 \) ways to choose four digits.
  - The probability that a player wins the large prize is \( \frac{1}{10000} = 0.0001 \).

- Solution2:
  - By sum rule, \( 9+9+9+9=36 \) ways to choose only three digits correctly.
  - By the product rule, there are \( 10 \times 10 \times 10 \times 10 = 10,000 \) ways to choose four digits.
  - The probability that a player wins the large prize is \( \frac{36}{10000} = 0.0036 \).
Probabilities of complements and unions of events

Let $E$ be an event in a sample space $S$. The probability of the event $\overline{E} = S - E$, the complementary event of $E$, is given by

$$p(\overline{E}) = 1 - p(E)$$

Example:
- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
- At least one bit is 0 = bits of length 10 - none of the 10 bits is 0
- $P = 1 - 1/2^{10} = 1023/1024$

Probabilities of complements and unions of events

Let $E_1$ and $E_2$ be events in the sample space $S$. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example:
- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?
- $E_1$: the integer selected at random is divisible by 2
- $E_2$: the integer selected at random is divisible by 5
- $P(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = 50/100 + 20/100 - 10/100 = 3/5$
Introduction

- If $S$ is a finite nonempty sample space of equally likely outcomes, and $E$ is an event, that is, a subset of $S$, then the probability of $E$ is $p(E) = \frac{|E|}{|S|}$.

- How to deal with experiments with outcomes are not equally likely?
  - A coin may be biased so that it comes up heads twice as often as tails.

- Conditional probability
  - Suppose that a fair coin is flipped four times, and the first time it comes up heads. Given this information, what is the probability that heads comes up three times?

- How to distinguish events are dependent or independent?
Assigning Probabilities

Let $S$ be the sample space of an experiment with a finite or countable number of outcomes. The probability $p(s)$ to each outcome $s$:

1. $0 \leq p(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} p(s) = 1$

When there are $n$ possible outcomes, $x_1, x_2, \ldots, x_n$,

1. $0 \leq p(x_i) \leq 1$ for $i = 1, 2, \ldots, n$
2. $\sum_{i=1}^{n} p(x_i) = 1$

Example

What probabilities should we assign to the outcomes $H$ (heads) and $T$ (tails) when a fair coin is flipped?

- $p(H) = p(T)$
- $p(H) + p(T) = 1$
- $p(H) = p(T) = 1/2$

What probabilities should we assign to these outcomes when the coin is biased so that heads comes up twice as often as tails?

- $p(H) = 2p(T)$
- $p(H) + p(T) = 1$
- $p(H) = 2/3, p(T) = 1/3$
Assigning Probabilities

Definition 1: Suppose that $S$ is a set with $n$ elements. The uniform distribution assigns the probability $1/n$ to each element of $S$.

- The experiment of selecting an element from a sample space with a uniform distribution is called selecting an element of $S$ at random.

Definition 2: The probability of the event $E$ is the sum of the probabilities of the outcomes in $E$. That is,

$$p(E) = \sum_{s \in E} p(s)$$

Example

Suppose that a die is biased so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

Solution:

- $E = \{1, 3, 5\}$
- $p(1) = p(2) = p(4) = p(5) = p(6) = 1/2p(3)$
- $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$
- $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$, $p(3) = 2/7$
- $p(E) = p(1) + p(3) + p(5) = 4/7$
Conditional Probability

Definition 3: Let $E$ and $F$ be events with $p(F) > 0$. The conditional probability of $E$ given $F$, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Example: A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Solution:

- $E$: the event that a bit string of length four contains at least two consecutive 0s
- $F$: the event that the first bit of a bit string of length four is a 0.
- The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals $p(E \mid F) = p(E \cap F)/p(F)$
- $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, $p(E \cap F) = 5/16$.
- Because there are eight bit strings of length four that start with a 0, $p(F) = 8/16 = 1/2$.
- Consequently, $p(E \mid F) = 5/16 / 1/2 = 5/8$

Conditional Probability

Example: What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl.

Solution:

- $E$: the event that a family with two children has two boys
- $F$: the event that a family with two children has at least one boy
- $E = \{BB\}, F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$.
- Because the four possibilities are equally likely, $p(F) = 3/4$ and $p(E \cap F) = 1/4$.
- $p(E \mid F) = p(E \cap F)/p(F) = \frac{1/4}{3/4} = 1/3$. 
The events $E$ and $F$ are independent $p(E \cap F) = p(E) \cdot p(F)$

if and only if

Example: suppose $E$ is the event that a randomly generated bit string of length four begins with a 1 and $F$ is the event that this bit string contains an even number of 1s. Are $E$ and $F$ independent, if the 16 bit strings of length four are equally likely?

- $0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111$
- $p(E) = p(F) = 8/16 = 1/2$
- $E \cap F = \{1111, 1100, 1010, 1001\}$
- $p(E \cap F) = 4/16 = 1/4$
- $p(E \cap F) = p(E) \cdot p(F)$
- $E$ and $F$ are independent

Example: Assume each of the four ways a family can have two children is equally likely. Are the events $E$, that a family with two children has two boys, and $F$, that a family with two children has at least one boy, independent?

Solution:

- Two children : BB, BG, GB, GG
- $E = \{BB\}$, then $p(E) = 1/4$.
- $F = \{BB, BG, GB\}$, then $p(F) = 3/4$
- $E \cap F = \{BB\}$, then $p(E \cap F) = 1/4$. But $p(E)p(F) = 1/4 \cdot 3/4 = 3/16$.
- Therefore $p(E \cap F) \neq p(E)p(F)$.
- so the events $E$ and $F$ are not independent.