Counting

Chapter 6
Chapter Summary

- 6.1 Discrete Probability
- 6.2 Probability Theory
- 6.3 Bayes’ Theorem
- 6.4 Expected value and Variance
Discrete probability

Section 6.1
Probability

- **Experiment**: a procedure that yields one of a given set of possible outcomes.
- **Sample space**: the set of possible outcomes of an experiment.
- **Event**: a subset of the sample space.
- If $S$ is a finite sample space of equally likely outcomes, and $E$ is an event, that is, a subset of $S$, then the probability of $E$ is

$$p(E) = \frac{|E|}{|S|}$$
Probability

- **Theorem 1** Let $E$ be an event in a sample space $S$. The probability of the event $\overline{E}$, the complementary event of $E$, is $p(\overline{E}) = 1 - p(E)$.
  - A sequence of 10 bits is randomly generated. $p$(at least 1 of these bits are 1s)?

- **Theorem 2** Let $E_1$ and $E_2$ be events in the sample space $S$. Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.
  - $p$(a positive integer not exceeding 100 is divisible by either 2 or 5)
- **Uniform distribution**: assigns the probability $1/n$ to each element of $S$.

**Theorem** If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space $S$, then

$$p\left(\bigcup_{i} E_i\right) = \sum_{i} p(E_i)$$

- The events $E$ and $F$ are independent if and only if $p(E \cap F) = p(E)p(F)$. 
Conditional Probability

Let $E$ and $F$ be events with $p(F) > 0$. The conditional probability of $E$ given $F$, denoted by $p(E|F)$, is defined as

$$p(E | F) = \frac{p(E \cap F)}{p(F)}$$
Bayes’ Theorem

Section 6.3
Why we need Bayes?

- How to assess the probability that a particular event occurs on the basis of partial evidence.

- The probability \( p(F) \) that an event \( F \) occurs and we have knowledge that an event \( E \) occurs. Then the conditional probability that \( F \) occurs given that \( E \) occurs, \( p(F \mid E) \), is a more realistic estimate than \( p(F) \) that \( F \) occurs.
Example

- Selects a ball by first choosing one of the two boxes at random.
- Selects one of the balls in this box at random.
- If Bob selected a red ball, what is the probability that he selected a ball from the first box.
Solution

- $E$: a RED ball is selected
- $\overline{E}$: a GREEN ball is selected
- $F$: a ball is selected from the first box
- $\overline{F}$: a ball is selected from the second box

\[
p(F \mid E) = \frac{p(F \cap E)}{p(E)}
\]

\[
p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{7}{9}, \quad p(F) = p(\overline{F}) = \frac{1}{2}
\]

\[
p(E \cap F) = \frac{7}{18}
\]

\[
p(E \mid \overline{F}) = \frac{p(E \cap \overline{F})}{p(F)} = \frac{3}{7}
\]

\[
p(E \cap \overline{F}) = \frac{3}{14}
\]

$E = (E \cap F) \cup (E \cap \overline{F})$

\[
P(E) = p(E \cap F) + p(E \cap \overline{F}) = \frac{38}{63}
\]

\[
p(F \mid E) = \frac{p(F \cap E)}{p(E)} = \frac{49}{76} \approx 0.645
\]
Bayes’ Theorem

Suppose that $E$ and $F$ are events from a sample space $S$ such that $p(E) \neq 0$ and $p(F) \neq 0$. Then

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}$$

Note: $F$ and $\overline{F}$ are mutually exclusive and cover the entire sample space $S$, that is $F \cup \overline{F} = S$. 

![Diagram showing the relationship between $E$, $F$, and $\overline{F}$]
Example

- Suppose one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99% of the time when given to someone with the disease; it is correct 99.5% of the time when given to someone who does not have the disease.
  - Only 0.2% of people who test positive actually have the disease.
- \( P(\text{someone who tests negative for the disease does not have the disease})? \)
- \( F: \text{a person has the disease} \)
- \( E: \text{a person tests positive} \)

\[
p(F | E) = \frac{p(E | F) p(F)}{p(E | F) p(F) + p(E | F^c) p(F^c)}
\]

\[
= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.005 \cdot 0.99999} \approx 0.002
\]

Only 0.2% of people who test positive actually have the disease.
Suppose one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99% of the time when given to someone with the disease; it is correct 99.5% of the time when given to someone who does not have the disease.

\[ P(\text{someone who test positive actually have the disease})? \]
\[ 99.99999\% \text{ of people who test negative really do not have the disease.} \]

F: a person has the disease
E: a person tests positive

\[
p(F \mid \overline{E}) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}
\]

\[
= \frac{0.995 \cdot 0.99999}{0.995 \cdot 0.99999 + 0.01 \cdot 0.00001} \approx 0.99999999
\]
Expected value and Variance

Section 5.5
Expected value

- The expected value of a random variable is the sum over all elements in a sample space of the product of the probability of the element and the value of the random variable at this element.

- The weighted average of the values of a random variable

\[
E(X) = \sum_{s \in S} p(s)X(s)
\]

Example

Let \(X\) be the number that comes up when a die is rolled. What is the expected value of \(X\)?

\[
E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{7}{2}
\]
Bernoulli Trials

- Bernoulli trial: an experiment whose outcome is random and can be either of two possible outcomes, "success" and "failure". In practice it refers to a single experiment which can have one of two possible outcomes.

- **THEOREM 2** The expected number of successes when \( n \) independent Bernoulli trials are performed, where \( p \) is the probability of success on each trial, is \( np \).
Linearity of Expectations

- If $X_i$ ($i=1, 2, \ldots, n$) are random variables on $S$, and if $a$ and $b$ are real numbers, then
  
  i. $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$
  
  ii. $E(aX + b) = aE(X) + b$
Example

Find the expected value of the sum of the numbers that appear when a pair of fair dice is rolled.

\[ E(X_1) = E(X_2) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2} \]
\[ E(X_1 + X_2) = E(X_1) + E(X_2) = 7 \]

\[
\begin{align*}
X((1,1)) &= 2 \\
X((1,2)) &= X((2,1)) = 3 \\
X((1,3)) &= X((2,2)) = X((3,1)) = 4 \\
X((1,4)) &= X((2,3)) = X((3,2)) = X((4,1)) = 5 \\
X((1,5)) &= X((2,4)) = X((3,3)) = X((4,2)) = X((5,1)) = 6 \\
X((1,6)) &= X((2,5)) = X((3,4)) = X((4,3)) = X((5,2)) = X((6,1)) = 7 \\
X((2,6)) &= X((3,5)) = X((4,4)) = X((5,3)) = X((6,2)) = 8 \\
X((3,6)) &= X((4,5)) = X((5,4)) = X((6,3)) = 9 \\
X((4,6)) &= X((5,5)) = X((6,4)) = 10 \\
X((5,6)) &= X((6,5)) = 11 \\
X((6,6)) &= 12
\end{align*}
\]
Example

A new employee checks the hats of $n$ people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the expected number of hats that are returned correctly?

$X=X_1+\ldots+X_n$: the number of people who receive the correct hat

$X_i=1$: $i$th person receives the correct hat

$X_i=0$: otherwise

$E(X)=E(X_1)+\ldots+E(X_n)$

$p(x_i=1)=1/n,$

$E(X_i)=1*p(x_i=1)+0*p(x_i=0)=1/n$

- The number of people who receive the correct hat is 1
Geometric Distribution

- a random variable with infinitely many possible outcomes

Example

Suppose that the probability that a coin comes up tails is \( p \). This coin is flipped repeatedly until it comes up tails. What is the expected number of flips until this coin comes up tails?

Sample space: \( \{T, HT, HHT, HHHT, HHHHT, \ldots\} \)

\( X \): the random variable equal to the number of flips

\[
E(X) = \sum_{n=1}^{\infty} n \cdot p(X = n)
\]

\[
= \sum_{n=1}^{\infty} n(1 - p)^{n-1} p
\]

\[
= p \sum_{n=1}^{\infty} n(1 - p)^{n-1}
\]

\[
= p \cdot \frac{1}{p^2} = \frac{1}{p}
\]
Geometric Distribution

A random variable $X$ has a geometric distribution with parameter $p$ if $p(X=k) = (1-p)^{k-1}p$ for $k=1, 2, 3, \ldots$

- Used to study the time required before a particular event happens.
- Time required before we find an object with a certain property.
- Number of attempts before an experiment succeeds.
- Number of times a product can be used before it fails.

- If the random variable $X$ has the geometric distribution with parameter $p$, then $E(X) = 1/p$. 
Geometric Distribution

A random variable \( X \) has a geometric distribution with parameter \( p \) if

\[
p(X=k) = (1-p)^{k-1}p \quad \text{for } k=1, 2, 3, \ldots
\]

- Used to study the time required before a particular event happens.
- Time required before we find an object with a certain property.
- Number of attempts before an experiment succeeds.
- Number of times a product can be used before it fails.

If the random variable \( X \) has the geometric distribution with parameter \( p \), then \( E(X) = \frac{1}{p} \).
Variance and Standard Deviation

How widely the values of a random variable are distributed?

- Let $X$ be a random variable on a sample space $S$. The variance of $X$, denoted by $V(X)$, is
  $$V(X) = \sum_{s\in S} (X(s) - E(X))^2 p(s).$$

- The standard deviation of $X$, denoted by $\delta(X)$, is defined to be $\sqrt{V(X)}$.

- If $X$ is a random variable on a sample space $S$, then $V(X) = E(X^2) - E(X)^2$.

- If $X$ and $Y$ are two independent random variables on a sample space $S$, then $V(X+Y) = V(X) + V(Y)$. Furthermore, if $X_i$ ($i=1, 2, \ldots, n$) are pairwise independent random variables on $S$, then $V(X_1+X_2+\ldots+X_n) = V(X_1) + V(X_2) + \ldots + V(X_n)$. 
Find the variance of a die.

A perfect die, when thrown, has expected value of 
\[
(1 + 2 + 3 + 4 + 5 + 6) / 6 = 3.5.
\]

Its variance is 
\[
(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) / 6 = 17.5/6 ≈ 2.9.
\]
Example

- The variance of the random variable $X$ whose value is the number of heads when tossing a coin twice.

- If a coin is tossed twice, the number of heads is: 0 with probability 0.25, 1 with probability 0.5 and 2 with probability 0.25. Thus the expected value of the number of heads is $0.25 \times 0 + 0.5 \times 1 + 0.25 \times 2 = 1$, and the variance is $0.25 \times (0 - 1)^2 + 0.5 \times (1 - 1)^2 + 0.25 \times (2 - 1)^2 = 0.25 + 0 + 0.25 = 0.5$. 