Maximal Independent Set and Minimum Connected Dominating Set in Unit Disk Graphs

Presented by Yingshu Li
yili@cs.umn.edu

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Introduction

A Unit Disk Graph (UDG) is an intersection graphs of circles of unit radius in the plane. An edge exists between two nodes $u$ and $v$ if and only if $|uv| \leq 1$.

Maximal Independent Set (MIS) is a maximal set of pair-wise non-adjacent vertices.
Introduction

A connected dominating set (CDS) of a graph is a subset of the nodes such that it forms a dominating set in the graph and the subgraph induced is connected. Computing a minimum CDS is NP-hard.

Alzoubi and Wan's algorithm

Construct a rooted spanning tree from the original network topology.

Color each node to be black or grey based on its rank (level, ID). The node with the lowest rank marks itself black. All the black nodes form an MIS.
Introduction

Alzoubi and Wan’s algorithm

Connect the nodes in the MIS to form a CDS.

Performance Ratio = 8

Introduction

- \( PR = \frac{|CDS|}{|\text{opt}|} \)
- \( PR \) is determined by
  - How large an MIS is compared to a MCDS
  - How many vertices are required to connect an MIS
- \( PR = 8 \) (2002)

Preliminaries

- \( N(x) \): neighbor area of \( x \), a unit disk centered at \( x \)
- \( N(G) \): the union of all neighbor areas of its vertices.
- \( u \) and \( v \) are adjacent if \( |uv| < 1 \) and independent if \( |uv| > 1 \)

Introduction

- It has been proved that
  \( \text{MIS}(G) \leq 4\text{MCDS}(G) + 1 \)
- In this paper
  \( \text{MIS}(G) \leq 3.8\text{MCDS}(G) + 1.2 \)
Preliminaries

Lemma 1 The neighbor area of a vertex contains at most five independent vertices.

\[ \angle x_i x_j x_k > 60^\circ \]

\[ k \leq 5 \]

Unit arc-triangle

Every two points in a Unit arc-triangle have distance at most 1.

Main Results

- **Lemma 2** A unit arc-triangle cannot contain two independent vertices.

- **Lemma 3** The neighbor area of two adjacent vertices contains at most 8 independent vertices.

- **Lemma 4** For any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.

- **Lemma 5** Every tree \( T \) with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.
Theorem For any unit disk graph \( G \), the size of a maximal independent set is at most \( 3.8 \text{cds}(G) + 1.2 \) where \( \text{cds}(G) \) is the size of a minimum connected dominating set.

Corollary

The best performance ratio of the current approximation algorithms for the MCDS is reduced from 8 to 7.8.

Lemma 4 For any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.
Theorem

\[ |T| \geq 3 \]

A non-leaf vertex \( v \) is adjacent to at most one non-leaf vertex \( u \), or \( v \) is adjacent to a leaf \( u \).

\( x_1, \ldots, x_k \) \((k \leq 4)\): other neighbors of \( v \)

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**Lemma 5** Every tree \( T \) with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.

**Lemma 3** The neighbor area of two adjacent vertices contains at most eight independent vertices.

**Lemma 1** The neighbor area of a vertex contains at most five independent vertices.

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**Theorem**

The neighbor area of \( T-\{v, x_1, \ldots, x_k\} \) contains at most \( 3.8(|T|-k-1)+1.2=\alpha \) independent vertices.

The neighbor area of \( T \) contains at most \( \alpha+7+4(k-1) = 3.8|T|+1.2+0.2(k-4) \leq 3.8|T|+1.2 \) independent vertices.

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**Discussion**

**Conjecture:**

The neighbor area of a 4-star subgraph in a unit disk graph contains at most twenty independent vertices.

If this is true, Theorem 1 can be improved from 3.8 to 3.6.
Lemma 3

The neighbor area of two adjacent vertices contains at most 8 independent vertices.

Lemma 1

The neighbor area of a vertex contains at most five independent vertices.
Lemma 3

- $a_0, a_1, \ldots, a_4$ lie counter-clockwisely in $N(u)$
- $a_0, a_5, \ldots, a_8$ lie counter-clockwisely in $N(v)$
- Let $u_{bi}$ be the radius containing $a_i$ for $i=2, \ldots, 4$ and $v_{bi}$ be the radius containing $a_i$ for $i=5, \ldots, 8$.
- Draw four unit arc-triangles $u_{b2}c_2$, $u_{b3}c_3$, $v_{b6}c_6$, $v_{b7}c_7$.
- $a_1, a_4, a_5, a_8$ must lie in the four small dark areas.

Lemma 2

A unit arc-triangle cannot contain two independent vertices.

Lemma 3

- $\angle b_2u_{b1} > 60^\circ$  
- $\angle c_3u_{b3} = \angle b_1u_{c3} = 60^\circ$  
- $\angle c_3u_{v} > 180^\circ$  
- $\angle c_6v_{c3} > 180^\circ$  
- $\angle c_3u_{v} + \angle c_6v_{c3} > 180^\circ$  
- $\angle c_2u_{c7} + \angle c_5v_{c7} > 180^\circ$  
- $\angle c_2u_{c7} + \angle c_5v_{c7} = 180^\circ$  
- $|c_2c_7| = |uv| \leq 1$

The distance between two points in $xc_2d_2$ and $xc_7d_7$ cannot exceed $\max(|c_2c_7|, |c_2d_7|, |d_2c_7|, |d_2d_7|)$.

$|d_2d_7| \leq \max(|c_2d_7|, |d_2c_7|) \leq 1$
Lemma 3

\[ |uv| = |vb_7| = |b_7d_7| = |d_7u| = 1 \]

\[ uvb_7d_7 \text{ and } c_2d_7b_7c_7 \text{ are parallelograms.} \]

\[ |c_2d_7| = |c_7b_7| = 1 \]

\[ |c_2d_7| \leq |d_2c_7| \leq 1 \]

Lemma 4

For any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.

Lemma 4

\[ T \text{ is a minimum spanning tree:} \]

- Two edges meeting at a vertex form an angle of at least 60°. Every vertex in \( T \) has degree at most 6.

- If two edges form an angle of exactly 60°, then they have the same length.
Lemma 5

Every tree $T$ with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.

Every leaf of $T'$ is a non-leaf vertex of $T$ which is adjacent to at most one non-leaf vertex.