The Basics of Counting

Section 6.1

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution:

- By the product rule,
- there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ different possible license plates.

<table>
<thead>
<tr>
<th>26 choices</th>
<th>10 choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>for each letter</td>
<td>for each digit</td>
</tr>
</tbody>
</table>
Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

\[ 26 + 26 \cdot 10 = 286 \]

Subtraction Rule

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

\[ 2^7 = 128 \]
\[ 2^6 = 64 \]
\[ 2^5 = 32 \]

\[ 128 + 64 - 32 = 160 \]
33. How many strings of eight English letters are there:
   a) that contain no vowels, if letters can be repeated?
      \[21^8\]
   b) that contain no vowels, if letters cannot be repeated?
      \[21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14\]
   c) that start with a vowel, if letters can be repeated?
      \[5 \times 21^8\]
   d) that start with a vowel, if letters cannot be repeated?
      \[5 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19\]
   e) that contain at least one vowel, if letters can be repeated?
      \[26^8 - 21^8\]
   f) that contain exactly one vowel, if letters can be repeated?
      \[8 \times 5 \times 21^7\]
   g) that start with X and contain at least one vowel, if letters can be repeated?
      \[1 \times 26^7 - 1 \times 21^7\]
   h) that start and end with X and contain at least one vowel, if letters can be repeated?
      \[1 \times 26^6 \times 1 - 1 \times 21^6 \times 1\]

The Pigeonhole Principle

Section 6.2
The Generalized Pigeonhole Principle

Example:

- a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- b) How many must be selected to guarantee that at least three hearts are selected?

Solution:

- a) Assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least \( \lceil \frac{N}{4} \rceil \) cards. At least three cards of one suit are selected if \( \lceil \frac{N}{4} \rceil \geq 3 \). The smallest integer \( N \) such that \( \lceil \frac{N}{4} \rceil \geq 3 \) is \( N = 2 \cdot 4 + 1 = 9 \).
- b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)
Solving Counting Problems by Counting Permutations

Example: How many permutations of the letters ABCDEFGH contain the string ABC?

Solution: We solve this problem by counting the permutations of six objects, ABC, D, E, F, G, and H.

\[ P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \]

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

\[ C(10, 5) = \frac{10!}{5!5!} = 252 \]
Permutations with Repetition

- Example: How many strings of length $r$ can be formed from the uppercase letters of the English alphabet if repetition is allowed?

- Solution: The number of such strings is $26^r$, which is the number of $r$-permutations of a set with 26 elements.

Theorem 1: The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$. 
Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: $1, $2, $5, $10, $20, $50, and $100?

Solution:

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.

This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11, 5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition Allowed?</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-permutations</td>
<td>No</td>
<td>$\frac{n!}{(n-r)!}$</td>
</tr>
<tr>
<td>r-combinations</td>
<td>No</td>
<td>$\frac{n!}{r!\ (n-r)!}$</td>
</tr>
<tr>
<td>r-permutations</td>
<td>Yes</td>
<td>$n^r$</td>
</tr>
<tr>
<td>r-combinations</td>
<td>Yes</td>
<td>$\frac{(n+r-1)!}{r!\ (n-1)!}$</td>
</tr>
</tbody>
</table>

TABLE 1 Combinations and Permutations With and Without Repetition.
Permutations with Indistinguishable Objects

- Example: How many different strings can be made by reordering the letters of the word SUCCESS.

- Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.
  - The three Ss can be placed in \( C(7,3) \) different ways, leaving four positions free.
  - The two Cs can be placed in \( C(4,2) \) different ways, leaving two positions free.
  - The U can be placed in \( C(2,1) \) different ways, leaving one position free.
  - The E can be placed in \( C(1,1) \) way.

- By the product rule, the number of different strings is:

\[
C(7, 3)C(4, 2)C(2, 1)C(1, 1) = \frac{7!}{3!} \cdot \frac{4!}{2!} \cdot \frac{2!}{1!} \cdot \frac{1!}{1!} = \frac{7!}{3!2!1!1!} = 420.
\]