A Permutation Network

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ABSTRACT. In this paper the construction of a switching network capable of $n!$-permutation of its n input terminals to its n output terminals is described. The building blocks for this network are binary cells capable of permuting their two input terminals to their two output terminals.

The number of cells used by the network is $\langle n \cdot \log_2 n - n + 1 \rangle = \sum_{k=1}^{n} \langle \log_2 k \rangle$. It could be argued that for such a network this number of cells is a lower bound, by noting that binary decision trees in the network can resolve individual terminal assignments only and not the partitioning of the permutation set itself which requires only $\langle \log_2 n! \rangle = \langle \sum_{k=1}^{n} \log_2 k \rangle$ binary decisions.

An algorithm is also given for the setting of the binary cells in the network according to any specified permutation.

KEY WORDS AND PHRASES: permutation, signal set, network, network design, flipflop

CR CATEGORIES: 6.1, 6.39, 6.9

In many applications of computer design it is desirable to efficiently construct a network which permutes a set of signals. Let the "elementary cell" be the basic building block of such a network, which by itself is a permutation network on two inputs and presumably can be constructed using a single flipflop (see Figure 1). Clearly, the lower bound on the number of elementary cells required for a permutation network on N signals is $\log_2 (N!)$.

We describe a construction which utilizes $F(N)$ cells, where N is a power of two, such that

$$\lim_{N \to \infty} \left(\frac{F(N)}{\log_2 (N!)} \right) = 1.$$

Definition 1. $u_1, u_2, \cdots, u_N$ are the N inlets to the permutation networks.

Definition 2. $v_1, v_2, \cdots, v_N$ are the N outlets from the permutation network.

Definition 3. $\{\pi(u_i) \to v_j : i, j = 1, \cdots, N\}$ is a permutation assigning u_i to some v_j for all i and j.

Definition 4. X_i is a variable ranging over the pair (u_{2i}, u_{2i-1}) for $i = 1, \cdots, (N/2)$.

Definition 5. Y_j is a variable ranging over the pair (v_{2j}, v_{2j-1}) for $j = 1, \cdots, (N/2)$.

THEOREM 1. *There exists a permutation* $\{\pi(X_i) \to Y_j : i, j = 1, \cdots, (N/2)\}$ *under the permutation* $\{\pi(u_i) \to v_j : i, j = 1, \cdots, N\}$.

PROOF. Let $K_i = \pi(X_i) \to Y_j$, under $\{\pi(u_i) \to v_j : i, j = 1, \cdots, N\}$. That is: K_i is a mapping of some u, $u \in X_i$, onto some v, $v \in Y_j$. For each K_i there

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FIG. 1. a, reset state; b, set state

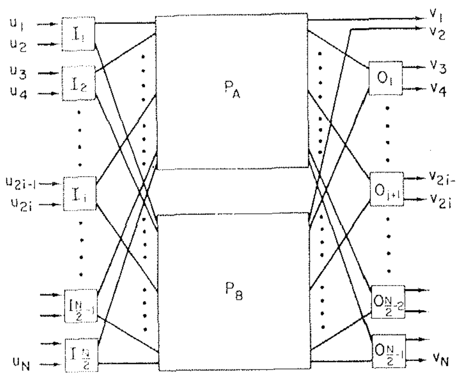

FIG. 2

are at most two such mappings where the two elements of X_i do not map onto t elements of a single Y_j. Thus the set $\{K_i\}$ contains at most two elements. N under the permutation $\{\pi(u_i) \rightarrow v_j : i, j : 1, \cdots, N\}$, no two elements of $\{u_i : i$ $1, \cdots, N\}$ are mapped onto a single element of $\{v_j : j = 1, \cdots, N\}$ so that $| \bigcup K_i$ $N/2, 1 \leq i \leq N/2$. But $| \bigcup X_i | = N/2, 1 \leq i \leq N/2$, so that by P. Hall's theor [1] on distinct representatives of sets (see Appendix), each X_i has a distinct rep sentative in $\bigcup K_i$, $1 \leq i \leq N/2$. Consequently there is a set $\{\pi(X_i) \rightarrow Y_i : i, j$ $\cdots, (N/2)\}$ which is one-to-one; hence a permutation. Q.E.D.

COROLLARY 1. *$\{\pi(X_i) \rightarrow Y_j\}$ is a subassignment of the assignment under permutation $\{\pi(u_i) \rightarrow v_j\}$ that involves exactly* one $u \in X_i$ *and* one $v \in Y_j$, *for i and j.*

COROLLARY 2. *v_K, an element of the set $\{v_i : i = 1, \cdots, N\}$, can be selected a trarily as a* fixed *representative of its corresponding Y_j, where $v_K \in Y_j$ for $\{\pi(X_i)$ $Y_j\}$ under all the permutations of $\{\pi(u_i) \rightarrow v_j : i, j = 1, \cdots, N\}$.*

PROOF. Since

$$| \bigcup K_i | = N/2, \qquad 1 \leq i \leq N/2,$$

there are always two sets of representatives which exhaust $\{u_i\}$ and $\{v_j\}$. Thus, any given permutation, we select the set that includes v_K.

Network Construction

In Figure 2, $I_1, \cdots, I_{N/2}$ and $O_1, \cdots, O_{N/2-1}$ represent elementary cells, and P_B represent permutation networks on $N/2$ inputs each. By Theorem 1 and Co lary 1, the subassignment $\{\pi(X_i) \rightarrow Y_j\}$ can be made through P_A so that eac and O_j has a single link to P_A and a single link to P_B for all i and j.

By Corollary 2 we can fix one Y variable, i.e., eliminate one cell, and we cho to eliminate the cell associated with v_1 and v_2. Thus Figure 2 is a permutation r work decomposed into $N - 1$ elementary cells and two permutation networks

$N/2$ inputs each. Clearly the process of decomposition can be continued until the complete network is decomposed into linked elementary cells.

For $N = 2^r$ there are r such decompositions, which result in the following number of elementary cells:

$$\begin{aligned} F(N) &= (2^r - 1) + 2(2^{r-1} - 1) + 2^2(2^{r-2} - 1) + \cdots + 2^{r-1}(2 - 1) \\ &= r2^r - (1 + 2' + 2^2 + 2^3 + \cdots + 2^{r-1}) \\ &= r2^r - (2^r - 1) = r2^r - 2^r + 1, \\ F(N) &= N \log_2 N - N + 1. \end{aligned}$$

A constructive proof[1] showing that the network of Figure 2 is indeed a permutation network is as follows.

Consider a network as in Figure 2 where the links between P_A, P_B and the input-output cells are omitted. Now let it be desired to realize a given arbitrary permutation by establishing the links one by one as dictated by the permutation assignment.

Start with v_1 and establish a link through P_A to some u_i through its corresponding I. Let state b of Figure 1 be the set state. Set I if u is even. Proceed next with the second u associated with this I and establish a link through P_B to its corresponding v through the O associated with it. Set this O if v is even. Repeat the process until all input-output pairs have been matched and the appropriate input and output cells have been set.

Note that in case a specific permutation involves more than a single cycle, a link will be established to v_2 via P_B before all assignments are made In this case any inlet or outlet terminal not yet linked can be used as a starting point.

Now, since by construction P_A and P_B are each associated with exactly $N/2$ inputs and $N/2$ outputs, and since by assumption P_A and P_B are permutation networks, the assignment $\{u_i \to v_j : i, j = 1, \cdots, N\}$ is complete and the link pattern is as in Figure 2. Q.E.D.

An algorithm can now be established for setting the decomposed cells of a network as well as the permutation required of P_A and P_B at any decomposition stage.

Algorithm

1. Express the given permutation to be realized by the network in an $N \times N$ permutation matrix.
2. Rewrite the $N \times N$ permutation matrix as an $(N/2) \times (N/2)$ partition matrix by merging coordinates corresponding to each common input or output cell.
3. Decompose the partition matrix into the sum of two $(N/2) \times (N/2)$ permutation matrices by letting v_1 be a coordinate in one (corresponding to P_A) and letting v_2 be a coordinate in the other (corresponding to P_B).
4. Identify the entries in the $N \times N$ matrix which correspond to the entries in the matrix of P_A.
5. Identify the coordinates in the $N \times N$ matrix which correspond to the entries identified in step 4.
6. Identify the cells associated with the coordinates identified in step 5.
7. Set (that is, put in state b, Figure 1) any cell identified in step 6 which is associated with an even coordinate identified in step 5.

[1] This proof is due to H. Stone.

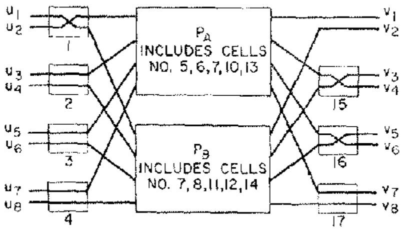

FIG. 3

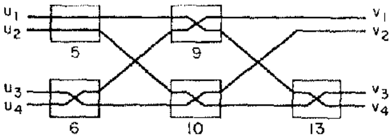

FIG. 4

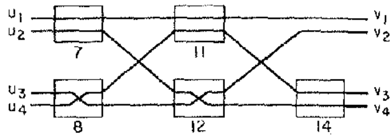

FIG. 5

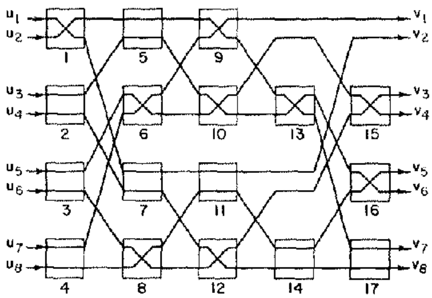

FIG. 6

Example

Let $N = 8$ and the required permutation $\binom{12345678}{27684315}$. The permutation matrix is

u \ v	1	2	3	4	5	6	7	8
1		1						
2							(1)	
3						(1)		
4								1
5				(1)				
6			1					
7	(1)							
8					1			

The partition matrix is

	1,2	Y_1	Y_2	Y_3
X_1	1			1
X_2			1	1
X_3		2		
X_4	1		1	

The two corresponding permutation matrices are

in \ out	1	Y_1	Y_2	Y_3
X_1				1
X_2			1	
X_3		1		
X_4	1			

P_1

in \ out	2	Y_1	Y_2	Y_3
X_1	1			
X_2				1
X_3		1		
X_4			1	

P_2

where P_1 corresponds to the permutation required by P_A, and P_2 corresponds to the permutation required of P_B. The entries in the 8×8 matrix corresponding to P_A are circled. Thus since u_2, v_4, and v_6 are even, we set the cells 1, 15, and 16 in Figure 3.

We are left with the task of decomposing and setting the permutation networks P_A and P_B. The permutation and partition matrices for P_A are

in \ out	1	2	3	4
1				(1)
2			1	
3		1		
4	(1)			

	12	Y_1
X_1		1
X_2	1	

The corresponding permutation networks P_1 and P_2 of P_A are

in \ out	1	Y_1
X_1		1
X_2	1	

P_1

in \ out	2	Y_1
X_1		1
X_2	1	

P_2

Since the even coordinates in the 4×4 P_A matrix identified with P_1 are u_4 and v_1, we set cells 6 and 13 in Figure 4. P_1 and P_2 also represent the permutation equired of cells 9 and 10 respectively in Figure 4, which call for setting these cells.

By following a similar procedure we arrive at Figure 5, which represents the net- ork required for the realization of P_B.

Figure 6 represents the complete network which realizes the required permutation.

ppendix

HALL'S THEOREM. *Let A be any set, and let $A_1, A_2, \cdots, A_r$ be any r subsets of A. s a necessary and sufficient condition that there exists a set of distinct representatives a, $\cdots, a_r$ of $A_1, \cdots, A_r$, i.e., elements $a_1, \cdots, a_r$ of A such that $a_i \in A_i$, $i = 1, \cdots, r$; $\neq a_j$ for $i \neq j$, it is necessary and sufficient that for each K in the range $1 \leq K \leq r$ e union of any K of the sets $A_1, \cdots, A_r$ have at least K elements.*

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